Comment on "Soliton ratchets induced by excitation of internal modes"

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Very recently Willis *et al.* [Phys. Rev. E **69**, 056612 (2004)] have used a collective variable theory to explain the appearance of a nonzero energy current in an ac-driven, damped sine-Gordon equation. In this Comment, we prove rigorously that the time-averaged energy current in an ac-driven nonlinear Klein-Gordon system is strictly zero.

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Recently several papers have been published trying to understand soliton ratchets (see, for example, Refs. [1–6] and for a recent review, Chap. 9 in Ref. [7], pp. 343–364). This phenomenon is a generalization of the ratchet effect [8] to spatially extended systems, and manifests as a unidirectional motion of a soliton induced by zero-average forces. A paradigmatic example is the driven, damped nonlinear Klein-Gordon equation

$$\phi_{,tt}(x,t) - \phi_{,xx}(x,t) = -U'[\phi(x,t)] + f(t) - \beta\phi_{,t}(x,t), \quad (1)$$

where $g_{,z}=\partial g/\partial z$, $f(t)$ is a periodic field with period T and zero time average [i.e., $1/T\int_0^T dt f(t)=0$], $\beta>0$ is the dissipation parameter determining the inverse relaxation time in the system, and $U'(z)$ is the derivative with respect to z of the potential $U(z)$. In this Comment, we will assume that the potential $U(z)$ is periodic with period λ, and presents minima at $z_j=z_0+j\lambda$, with $j\in\mathbb{Z}$. The ac-driven, damped sine-Gordon equation considered in Ref. [6] is a particular case of this more general problem, with $U(z)=1-\cos(z)$ and

$$f(t) = -[\epsilon_1\cos(\omega t) + \epsilon_2\cos(2\omega t+\theta)]. \quad (2)$$

To fully specify the mathematical problem, the partial differential equation (1) must be amended by both initial conditions for $\phi(x,0)$ and $\phi_{,t}(x,0)$, and boundary conditions for $\lim_{x\to\pm\infty}\phi(x,t)$. Several boundary conditions can be imposed to have a well-posed boundary value problem. For instance, in the absence of the periodic field $f(t)$, it is possible to choose the fixed boundary conditions: $\lim_{x\to+\infty}\phi(x,t)=z_l$ and $\lim_{x\to-\infty}\phi(x,t)=z_m$. In the presence of $f(t)$, the fixed boundary conditions become incompatible with Eq. (1), and they are usually replaced by the aperiodic boundary conditions:

$$\lim_{x\to+\infty}\phi(x,t) = \lim_{x\to-\infty}\phi(x,t) + \lambda Q, \quad (3)$$

$$\lim_{x\to+\infty}\phi_{,x}(x,t) = \lim_{x\to-\infty}\phi_{,x}(x,t), \quad (4)$$

where $Q\in\mathbb{Z}$ is the so-called topological charge. The discrete version of these aperiodic boundary conditions are also the most used in the numerical solution of Eq. (1) (see, for example, Refs. [1,5]).

It can be derived from the continuity equation that the energy current density generated by the field $\phi(x,t)$ in the absence of damping and external forcing is given by $j(x,t)=-\phi_{,t}(x,t)\phi_{,x}(x,t)$ and, consequently, the energy current reads

$$J(t) = -\int_{-\infty}^{+\infty} dx\,\phi_{,x}(x,t)\phi_{,t}(x,t). \quad (5)$$

The time-averaged energy current $\langle J\rangle$ is defined as the limit

$$\langle J\rangle = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau dt J(t). \quad (6)$$

In Ref. [1], it has been proved by symmetry considerations that a *necessary* condition for the appearance of a nonvanishing time-averaged energy current is that either the potential presents broken spatial symmetry, or the field $f(t)$ violates the symmetry property

$$f\left(t+\frac{T}{2}\right) = -f(t), \quad (7)$$

or both simultaneously. Following this idea, a collective variable approach has been developed in Ref. [6] for the ac-driven, damped sine-Gordon equation with a field of the form (2) that leads to a nonvanishing time-averaged energy current. The purpose of this Comment is to prove that the time-averaged energy current, $\langle J\rangle$, of a driven, damped nonlinear Klein-Gordon equation of the form (1) is necessarily zero.

To prove that $\langle J\rangle=0$, firstly we will obtain an ordinary differential equation for the energy current $J(t)$. In order to

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do that, we differentiate with respect to time Eq. (5), resulting

$$\dot{J}(t) = -\int_{-\infty}^{+\infty} dx[\phi_{,xt}(x,t)\phi_{,t}(x,t) + \phi_{,x}(x,t)\phi_{,tt}(x,t)]. \quad (8)$$

By making use of Eq. (1) in the second term on the right-hand side of the above expression, it is straightforward to write it in the form

$$\begin{aligned}\dot{J}(t) = &-\frac{1}{2}\left\{\lim_{x\to+\infty}[\phi_{,t}(x,t)]^2 - \lim_{x\to-\infty}[\phi_{,t}(x,t)]^2\right\} \\ &-\frac{1}{2}\left\{\lim_{x\to+\infty}[\phi_{,x}(x,t)]^2 - \lim_{x\to-\infty}[\phi_{,x}(x,t)]^2\right\} \\ &+\lim_{x\to+\infty}U[\phi(x,t)] - \lim_{x\to-\infty}U[\phi(x,t)] - \beta J(t) \\ &-\left[\lim_{x\to+\infty}\phi(x,t) - \lim_{x\to-\infty}\phi(x,t)\right]f(t). \end{aligned} \quad (9)$$

Differentiating Eq. (3) with respect to t, it is easy to see that the first term between the brace brackets on the right-hand side of Eq. (9) is equal to zero. From Eq. (4), it follows that the second term between the brace brackets on the right-hand side of Eq. (9) is also equal to zero. The two terms of Eq. (9) containing $U[\phi(x,t)]$ also cancel each other due to the boundary condition (3) and the periodicity of $U(z)$. Thus, from Eq. (3) we finally obtain

$$\dot{J}(t) = -\beta J(t) - \lambda Q f(t). \quad (10)$$

Notice that Eq. (10) is a direct consequence of Eq. (1) and the boundary conditions (3) and (4). Therefore, it is an *exact* result valid for any periodic potential $U(z)$ of the type described in the paragraph below Eq. (1), and any external field $f(t)$. Equation (10) appears in Ref. [6] as an *approximate* result obtained after neglecting the dressing due to phonons.

The general solution of Eq. (10) is

$$J(t) = J(0)e^{-\beta t} - \lambda Q\int_0^t dt' e^{-\beta(t-t')}f(t'), \quad (11)$$

and making use of the definition of the time-averaged energy current in Eq. (6), it results

$$\langle J\rangle = \lim_{\tau\to+\infty}\left\{\frac{J(0)}{\beta\tau}(1-e^{-\beta\tau}) - \frac{\lambda Q}{\beta\tau}\int_0^\tau dt\, f(t) + \frac{\lambda Q}{\beta\tau}\int_0^\tau dt\, f(t)e^{-\beta(\tau-t)}\right\}. \quad (12)$$

The first limit in the above expression is obviously equal to zero. The second one is also equal to zero as the external field is periodic with zero time average. The last integral appearing in Eq. (12) can be bounded using the fact that

$$\left|\int_0^\tau dt\, f(t)e^{-\beta(\tau-t)}\right| \leqslant \int_0^\tau dt|f(t)|e^{-\beta(\tau-t)} \leqslant \frac{f_m}{\beta}(1-e^{-\beta\tau}), \quad (13)$$

where f_m is an upper bound of $|f(t)|$ and, thus, the third limit in Eq. (12) is also equal to zero. We conclude that $\langle J\rangle = 0$ for any periodic potential $U(z)$ of the type described in the paragraph below Eq. (1), and any bounded, zero time-averaged periodic field $f(t)$.

It is important to emphasize that the result in this Comment is not applicable when the external field not only depends on t but also on x. In that case Eq. (10) cannot be obtained and, in principle, it is possible to observe a nonvanishing time-averaged energy current. A field of this kind has been considered in Ref. [1], where $f(x,t)=E(t)+\xi(x,t)$, with $E(t)$ being an ac field with zero mean and $\xi(x,t)$ a Gaussian white noise.

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